

# Reconstruction of Scalar Potentials in $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$ theory of gravity

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In this paper, we explore the nature of scalar field potential in  $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$  gravity using a well-motivated reconstruction scheme for flat FRW geometry. The beauty of this scheme lies in the assumption that the Hubble parameter can be expressed in terms of scalar field and vice versa. Firstly, we develop field equations in this gravity and present some general explicit forms of scalar field potential via this technique. In the first case, we take De Sitter universe model and construct some field potentials by taking different cases for coupling function. In the second case, we derive some field potentials using power law model in the presence of different matter sources like barotropic fluid, cosmological constant and Chaplygin gas for some coupling functions. From graphical analysis, it is concluded that using some specific values of the involved parameters, the reconstructed scalar field potentials are cosmologically viable in both cases.

**Keywords:** Scalar-tensor theory; Scalar field; Field potentials.

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## I. INTRODUCTION

The investigation about the possible causes of accelerated expansion of cosmos and nature of its missing mass and energy are some leading topics of this century. Numerous researchers are working over these lines and usually they proposed two ways to describe this accelerating expansion [1]. Some consider, general relativity (GR), as the right theory of gravity which present dark energy (DE) [2] as an easily conveyed, gradually changing cosmic fluid with negative pressure. This technique is referred as GR with modified matter sources. Other way is to modify the gravitational sector of GR [3]-[6]. The  $F(R)$  gravity, is one of the most attracting examples of modified gravity

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theories, that can be mapped into GR with extra scalar fields using an appropriate conformal transformation of metric [5]-[7]. In modern cosmology, scalar fields play an important role in explaining the nature of DE [2] and to drive inflation in the beginning of universe [8, 9]. In literature like [10], it is concluded that the history of cosmic expansion can be successfully discussed using scalar-tensor theories.

Many cosmological models and modified gravity theories with scalar fields, involve some general functions which cannot be derived easily from the basic theory. Then some questions frequently arise like how these particular functions should be chosen and what are physical reasons behind those particular choices. In this respect, the reconstruction technique is not a new concept, it has a long history to reconstruct the DE models. This technique allows one to find the form of scalar field potential as well as of scalar field for a specific choice of Hubble parameter in terms of scale factor or cosmic time. For a better understanding of this technique, we refer the readers to see the literature [38].

In scalar tensor theories, it is very necessary to investigate the nature of scalar field potential and its role in explaining DE and cosmic expansion history. In [39], the nature of scalar potential has been discussed for a minimally coupled scalar tensor theory using reconstruction technique. The reconstruction technique to explore the nature of field potentials for the models involving minimally coupling to scalar fields, two-field models and tachyon models has been given in literature [11]-[28]. The reconstruction of field potentials in the models involving non-minimally coupling of scalar fields to gravity was studied in [29]-[31]. Furthermore, the applications of this reconstruction technique in different gravity models and theories like the models containing non-minimally coupling of Yang-Mills fields [32], in the framework of local [37] and nonlocal gravity [36], in  $F(R)$  and Gauss-Bonnet gravity theories [21, 33, 34] and the  $F(T)$  gravity theory that involves torsion scalar  $T$  as a basic ingredient [35] are available in literature.

The models of gravity that are non-minimally coupled to scalar fields are of great interest in cosmology [40]-[47]. Particularly, the models having Hilbert-Einstein term in addition to the term relative to the Ricci scalar with squared scalar field were considered in quantum and inflationary cosmology [48, 49]. In [30], the reconstruction process has been studied for induced gravity ( $U(\phi) = \xi\phi^2$ , where  $\xi$  is an arbitrary constant). It is shown that for these cases, linearizing the differential equations to solve in reconstruction process, to derive potentials according to committed cosmological evolution. It is interesting to mention here that from this process, one can get explicit potentials which can reproduce the dynamics of flat (FRW) universe derived by different matter sources like barotropic and perfect fluids, Chaplygin gas [50], and modified Chaplygin gas

[30].

In this regard, Kamenshchik et al. [30] has used this approach to reconstruct the field potential in terms of scalar field for FRW universe in the framework of induced gravity. They discussed this procedure for different matter distributions and concluded that the corresponding cosmic evolution can be reproduced in these cases. In [51], the same authors used another technique known as super potential reconstruction technique for FRW model to reconstruct scalar field potentials in a non-minimally coupled scalar tensor gravity. They examined its nature for de-Sitter and barotropic models and discussed their cosmic evolution. Sharif and Waheed [52] studied the nature of scalar field potential for locally rotationally symmetric (LRS) Bianchi type I (BI) universe model in a general scalar-tensor theory via reconstruction technique and they concluded that the reconstructed potentials are viable on cosmological grounds. In a recent paper [54], we have discussed cosmological reconstruction and energy bounds in a new general  $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$  gravity. In this gravity, we have also studied the first and second laws of black hole thermodynamics for both equilibrium and non-equilibrium descriptions [55]. Being motivated from the literature, in the present paper, we examine the nature of field potential for flat FRW universe in  $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$  gravity by applying reconstruction procedure.

This paper is arranged in the following pattern. In the next section, we give a general overview of this procedure and derive the general form of scalar field potential for this theory. In section III, we derive field potentials for de-Sitter model by taking different choices for function  $f$ . Section IV is devoted to explore the form of field potential for a power law model with matter sources as barotropic fluid, cosmological constant and Chaplygin gas in separate cases. In the last section, we present a summary of all sections by highlighting the major achievements.

## II. BASIC FIELD EQUATIONS AND GENERAL SCALAR FIELD POTENTIAL

In this section, we present the basic formulations of the most general scalar-tensor gravity namely  $f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi)$  theory. The gravitational action for this theory is given as follows [53],

$$S_m = \int d^4x \sqrt{-g} \left[ \frac{1}{\kappa^2} (f(R, Y, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}) + V(\phi) \right], \quad (1)$$

where  $f$  is a general function depending upon the Ricci scalar  $R$ , the curvature invariant  $Y \equiv R_{\alpha\beta}R^{\alpha\beta}$  (where  $R_{\alpha\beta}$  is the Ricci tensor) and the scalar field symbolized by  $\phi$ . Further,  $\omega$  is a coupling function of scalar field  $\phi$ , the symbol  $V(\phi)$  corresponds to the scalar field potential and  $g$  is the determinant of metric tensor  $g_{\mu\nu}$  whereas  $\kappa^2$  represents the gravitational coupling constant.

The flat FRW spacetime with cosmic radius  $a$  is given by the following metric

$$ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2) . \quad (2)$$

For which, the quantities like scalar curvature  $R$  and Ricci invariant  $Y$  turn out to be

$$R = -6 \left( \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right), \quad Y = 6 \left( \frac{\dot{a}^4}{a^4} + \frac{2\dot{a}^2\ddot{a}}{a^3} + \frac{3\ddot{a}^2}{a^2} \right), \quad (3)$$

while  $\sqrt{-g} = a^3$ . The Friedmann equations constructed in [55] are

$$\begin{aligned} f(R, Y, \phi) + 6 \left( \dot{H} + 3H^2 \right) f_R + 6H \partial_t f_R + \omega(\phi) \dot{\phi}^2 + V(\phi) + 2 \left( \ddot{H} + 4H\ddot{H} + 6\dot{H}H^2 - 2H^4 \right) f_Y \\ + 12H \left( 3H^2 + 2\dot{H} \right) \partial_t f_Y = 0, \end{aligned} \quad (4)$$

and

$$\begin{aligned} f(R, Y, \phi) + \left( 10\dot{H} + 18H^2 \right) f_R + 4H \partial_t f_R + 2\partial_{tt} f_R - \omega(\phi) \dot{\phi}^2 - V(\phi) + \left( 8\ddot{H} + 40H\ddot{H} + 20H^2\dot{H} \right. \\ \left. + 32\dot{H}^2 - 36H^4 \right) f_Y + 8H \left( \dot{H} + 3H^2 \right) \partial_t f_Y + \left( 8\dot{H} + 12H^2 \right) \partial_{tt} f_Y = 0. \end{aligned} \quad (5)$$

The Klein-Gordon equation is

$$2\omega(\phi) \ddot{\phi} + \frac{d\omega}{d\phi} \dot{\phi}^2 + 6H\omega(\phi) \dot{\phi} + \frac{df}{d\phi} + \frac{dV}{d\phi} = 0. \quad (6)$$

From Eq. (4), we have

$$\begin{aligned} V(\phi) = -f - 6 \left( \dot{H} + 3H^2 \right) f_R - 6H \partial_t f_R - 2 \left( 3\ddot{H} + 4H\ddot{H} + 6H^2\dot{H} - 2H^4 \right) f_Y - 12H \left( 2\dot{H} \right. \\ \left. + 3H^2 \right) \partial_t f_Y - \omega(\phi) \dot{\phi}^2. \end{aligned} \quad (7)$$

It is more appropriate to consider all the functions dependent on “ $a$ ” instead of cosmic time “ $t$ ”,

$$\begin{aligned} V(\phi) = -f - 6 \left( 3H^2 + aHH' \right) f_R + 36H \left( a^2H^2H'' + a^2HH'^2 + 5aH^2H' \right) f_{RR} - 2 \left( a^3H^3H''' \right. \\ \left. + 4a^3H^2H'H'' + 7a^2H^3H'' + 8a^2H^2H'^2 + a^3HH'^3 + 11aH^3H' - 2H^4 \right) f_Y - 72H \left( 24a^2H^6H'' \right. \\ \left. + 34a^3H^5H'H'' + 96aH^6H' + 154a^2H^5H'^2 + 78a^3H^4H'^3 + 12a^4H^3H'^4 + 12a^4H^4H'^2H'' \right) f_{YY} \\ - 6aH^2\phi' f_{R\phi} - 12aH^2\phi' \left( 2aHH' + 3H^2 \right) f_{Y\phi} - 36H \left( 2a^2H^4H'' + 2a^3H^3H'H'' + 2aH^4H' \right. \\ \left. + 4a^2H^3H'^2 + 2a^3H^2H'^3 \right) f_{RY} - a^2H^2\omega(\phi)\phi'^2. \end{aligned} \quad (8)$$

By substituting the derivative of Eq. (7) in Eq. (6), we eliminate  $dV/d\phi$  and the resulting equation can be written as

$$\begin{aligned} 6H\ddot{\phi} f_{R\phi} + 12H \left( 2\dot{H} + 3H^2 \right) \ddot{\phi} f_{Y\phi} + 6 \left( 3H^2 + 2\dot{H} \right) \dot{\phi} f_{R\phi} + 2 \left( \ddot{H} + 16H\ddot{H} + 60H^2\dot{H} + 12\dot{H}^2 \right. \\ \left. - 2H^4 \right) \dot{\phi} f_{Y\phi} - 36H \left( \ddot{H} + 4H\dot{H} \right) \dot{\phi} f_{RR\phi} + 36H \left( 2H^2\ddot{H} + 2\dot{H}\ddot{H} \right) \dot{\phi} f_{RY\phi} + 72H \left( 24H^4\ddot{H} + 34 \right. \\ \left. \times H^2\dot{H}\ddot{H} + 12\dot{H}^2\ddot{H} + 96H^3\dot{H}^2 + 32H\dot{H}^3 + 72H^5\dot{H} \right) \dot{\phi} f_{YY\phi} - 6H\omega(\phi)\dot{\phi}^2 + 6H\dot{\phi}^2 f_{R\phi\phi} + 12H \end{aligned} \quad (9)$$

$$\begin{aligned}
& \times (2\dot{H} + 3H^2) \dot{\phi}^2 f_{Y\phi\phi} + 6(\ddot{H} + 6H\dot{H}) f_R + 2(\ddot{H} + 4H\ddot{H} + 4\dot{H}\ddot{H} + 12H\dot{H}^2 + 6H^2\ddot{H} - 8H^3 \\
& \times \dot{H}) f_Y - 36(H\ddot{H} + 4H^2\ddot{H} + \dot{H}\ddot{H} + 8H\dot{H}^2) f_{RR} + 36(2H^3\ddot{H} + 2H\dot{H}\ddot{H} + 6H^2\dot{H}\ddot{H} + 2\dot{H}^2\ddot{H} \\
& + 2H\ddot{H}^2) f_{RY} + 72(34H^3\dot{H}\ddot{H} + 24H^5\ddot{H} + 12H\dot{H}^2\ddot{H} + 312H^4\dot{H}\ddot{H} + 198H^2\dot{H}^2\ddot{H} + 24H\dot{H}\ddot{H}^2 \\
& + 34H^3\ddot{H}^2 + 12\dot{H}^3\ddot{H} + 72H^6\ddot{H} + 432H^5\dot{H}^2 + 384H^3\dot{H}^3 + 64H\dot{H}^4) f_{YY} = 0
\end{aligned} \tag{10}$$

In terms of “ $a$ ”, the above equation is written as A.1 given in the Appendix. The field equations involve five unknowns namely  $f$ ,  $a$ ,  $\phi$ ,  $\omega$  and  $V$ . Now we evaluate the scalar potential  $V$  for de-Sitter and power law models (in barotropic fluid, cosmological constant and in Chaplygin gas) by taking different choices of for the remaining unknowns.

### III. DE-SITTER MODELS

In cosmology, the dS-solutions are of great significance to explain the current cosmic epoch. In dS-model, the scale factor, the Hubble parameter and the Ricci tensor take the following form [56]

$$a(t) = a_0 e^{H_0 t}, \quad H = H_0, \quad R = 12H_0^2, \quad Y = 36H_0^4. \tag{11}$$

Here we are using  $\omega(\phi) = \omega_0 \phi^m$  and  $\phi(t) \sim a(t)^\beta$  [56].

#### A. $f(R, Y, \phi)$ Model

We have derived the general form of  $f(R, Y, \phi)$  for dS-model in [54], here we use this form to evaluate the scalar field potential. For this model, function  $f$  is defined as

$$f(R, Y, \phi) = \alpha_1 \alpha_2 \alpha_3 e^{\alpha_1 R} e^{\alpha_2 Y} \phi^{\gamma_1} + \gamma_2 \phi^{\gamma_3} + \gamma_4 \phi^{\gamma_5}, \tag{12}$$

where  $\alpha'_i$ s are constants of integration and

$$\begin{aligned}
\gamma_1 &= \frac{18\alpha_1\beta H_0^2 - 108\alpha_2\beta H_0^4 - 5 + 6\alpha_1 H_0^2 - 84\alpha_2 H_0^4}{6(\alpha_1\beta H_0^2 - 6\alpha_2\beta H_0^4)}, \quad \gamma_2 = \omega_0\beta^2 H_0^2, \quad \gamma_3 = m + 2, \\
\gamma_4 &= -2\kappa^2 \rho_0 a_0^{3(1+w)}, \quad \gamma_5 = -\frac{3}{\beta}.
\end{aligned} \tag{13}$$

- Case-I:  $\omega(\phi) = \omega_0 \phi^m$

Substituting model (12) into (A.1) and choosing  $m = \gamma_1 - 2$ , we have

$$\phi'' + \frac{4}{3a} \left( \frac{3\alpha_1 + 4\alpha_2 H_0^2}{\alpha_1 + 6\alpha_2 H_0^2} \right) \phi' + \left\{ \gamma_1 - 1 - \frac{\omega_0}{\alpha_1 \alpha_2 \alpha_3 \gamma_1 e^{\alpha_1 R} e^{\alpha_2 Y} (\alpha_1 + 6\alpha_2 H_0^2)} \right\} \frac{\phi'^2}{\phi} = 0. \tag{14}$$

For the sake of simplicity, we introduce a new variable  $\sigma = \phi'/\phi$ , thus (14) takes the following form:

$$\sigma' + \frac{4}{3a} \left( \frac{3\alpha_1 + 4\alpha_2 H_0^2}{\alpha_1 + 6\alpha_2 H_0^2} \right) \sigma + \left\{ \gamma_1 - \frac{\omega_0}{\alpha_1 \alpha_2 \alpha_3 \gamma_1 e^{\alpha_1 R} e^{\alpha_2 Y} (\alpha_1 + 6\alpha_2 H_0^2)} \right\} \sigma^2 = 0. \quad (15)$$

For further simplification, by introducing a new function

$$\sigma = \left\{ \gamma_1 - \frac{\omega_0}{\alpha_1 \alpha_2 \alpha_3 \gamma_1 e^{\alpha_1 R} e^{\alpha_2 Y} (\alpha_1 + 6\alpha_2 H_0^2)} \right\}^{-1} \frac{u'}{u} = A^{-1} \frac{u'}{u}, \quad (16)$$

we get a differential equation for  $u$  of the form:

$$u'' + \frac{4\xi}{3a} u' = 0, \quad (17)$$

where  $\xi = \left( \frac{3\alpha_1 + 4\alpha_2 H_0^2}{\alpha_1 + 6\alpha_2 H_0^2} \right)$ . It is easy to see, on comparing that

$$\phi = u^{1/A}. \quad (18)$$

Equations (8) and (12) lead to the following form of scalar potential:

$$\begin{aligned} V(\phi) = & -\alpha_1 \alpha_2 \alpha_3 e^{\alpha_1 R} e^{\alpha_2 Y} \phi^{\gamma_1} - \gamma_2 \phi^{\gamma_3} - \gamma_4 \phi^{\gamma_5} - 18 H_0^2 \alpha_1^2 \alpha_2 \alpha_3 e^{\alpha_1 R} e^{\alpha_2 Y} \phi^{\gamma_1} - 6 a H_0^2 \phi' \alpha_1^2 \alpha_2 \\ & \times \alpha_3 \gamma_1 e^{\alpha_1 R} e^{\alpha_2 Y} \phi^{\gamma_1-1} + 4 \alpha_1 \alpha_2^2 \alpha_3 H_0^4 e^{\alpha_1 R} e^{\alpha_2 Y} \phi^{\gamma_1} - 36 a H_0^4 \phi' \alpha_1 \alpha_2^2 \alpha_3 \gamma_1 e^{\alpha_1 R} e^{\alpha_2 Y} \phi^{\gamma_1-1} \\ & - \omega_0 a^2 H_0^2 \phi^{\gamma_1-2} \phi'^2. \end{aligned} \quad (19)$$

If  $\dot{\phi} \neq 0$ , all the assumptions regarding derivative of (7) are justifiable. The constant field must be discussed separately. First consider, if  $\phi$  is a non-zero constant, then  $f$ ,  $\omega$  and  $V$  are also all independent of time. This leads to the cosmological evolution which is occurred due to cosmological constant. Now we can rewrite Eq.(4) as follows

$$V = -f + 4H_0^4 f_Y - 18H_0^2 f_R. \quad (20)$$

Then, on substituting  $\dot{\phi} = 0$  and  $\dot{H} = 0$  into Eq.(6), we have

$$\frac{dV}{d\phi} = -\frac{df}{d\phi}. \quad (21)$$

Multiplying Eq.(20) by Eq.(21) and then integrating, we have

$$\begin{aligned} V^2 = & (1 - 4H_0^4 \alpha_2 + 18H_0^2 \alpha_1) \alpha_1^2 \alpha_2^2 \alpha_3^2 e^{24\alpha_1 H_0^2} e^{72\alpha_2 H_0^4} \phi^{2\gamma_1} + \frac{2\alpha_1 \alpha_2 \alpha_3 \gamma_2}{\gamma_1 + \gamma_3} (\gamma_1 + \gamma_3 - 4H_0^4 \alpha_2 \gamma_3 \\ & + 18H_0^2 \alpha_1 \gamma_3) e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} \phi^{\gamma_1 + \gamma_3} + \frac{2\alpha_1 \alpha_2 \alpha_3 \gamma_4}{\gamma_1 + \gamma_5} (\gamma_1 + \gamma_5 - 4H_0^4 \alpha_2 \gamma_5 + 18H_0^2 \alpha_1 \gamma_5) e^{12\alpha_1 H_0^2} \\ & e^{36\alpha_2 H_0^4} \phi^{\gamma_1 + \gamma_5} + \gamma_2^2 \phi^{2\gamma_3} + 2\gamma_2 \gamma_4 \phi^{\gamma_3 + \gamma_5} + \gamma_4^2 \phi^{2\gamma_5}. \end{aligned} \quad (22)$$

Now for the previous case, i.e., time dependent scalar field, the basic equation (17) is given by

$$a^2 u'' + \frac{4}{3} a \xi u' = 0, \quad (23)$$

whose general solution is

$$u(a) = c_1 + c_2 a^{-\xi/3}, \quad (24)$$

where  $c_1$  and  $c_2$  are integration constants. From (24), it can be written as

$$a = \left( \frac{c_2}{u - c_1} \right)^{3/\xi}, \quad (25)$$

and from (18), we have

$$\phi = u^{1/A} = \left( c_1 + c_2 a^{-\xi/3} \right)^{1/A}, \quad (26)$$

and inversely, it takes the form

$$a = \left( \frac{c_2}{\phi^A - c_1} \right)^{3/\xi}. \quad (27)$$

Another useful formula, in this respect, is

$$\phi' a = -\frac{\xi}{3A} \phi (1 - c_1 \phi^{-A}). \quad (28)$$

Inserting (26) and (28) into Eq.(19), we get

$$\begin{aligned} V(\phi) = & \left( -\alpha_1 \alpha_2 \alpha_3 e^{\alpha_1 R} e^{\alpha_2 Y} - 18 \alpha_1^2 \alpha_2 \alpha_3 H_0^2 e^{\alpha_1 R} e^{\alpha_2 Y} + \frac{2\xi}{A} \alpha_1^2 \alpha_2 \alpha_3 \gamma_1 H_0^2 e^{\alpha_1 R} e^{\alpha_2 Y} + 4 \alpha_1 \alpha_2^2 \alpha_3 \right. \\ & \times H_0^4 e^{\alpha_1 R} e^{\alpha_2 Y} + \frac{12\xi}{A} \alpha_1 \alpha_2^2 \alpha_3 \gamma_1 H_0^4 e^{\alpha_1 R} e^{\alpha_2 Y} - \frac{\omega_0 \xi^2}{9A^2} H_0^2 \left. \right) \phi^{\gamma_1} - \gamma_2 \phi^{\gamma_3} - \gamma_4 \phi^{\gamma_5} + \left( \frac{2\xi c_1}{A} \alpha_1^2 \alpha_2 \right. \\ & \times \alpha_3 \gamma_1 H_0^2 e^{\alpha_1 R} e^{\alpha_2 Y} - \frac{12\xi c_1}{A} \alpha_1 \alpha_2^2 \alpha_3 \gamma_1 H_0^4 e^{\alpha_1 R} e^{\alpha_2 Y} + \frac{2\omega_0 c_1 \xi^2}{9A^2} H_0^2 \left. \right) \phi^{\gamma_1 - A} - \frac{\omega_0 \xi^2 c_1^2}{9A^2} H_0^2 \phi^{\gamma_1 - 2A}. \end{aligned} \quad (29)$$

If we choose  $c_1 = 0$  in (26), we have  $\phi = c_2 a^{-\xi/3A}$  and using this in the above equation, we get the scalar potential in terms of scale factor as follows

$$\begin{aligned} V(a) = & \left( -\alpha_1 \alpha_2 \alpha_3 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} - 18 \alpha_1^2 \alpha_2 \alpha_3 H_0^2 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} + \frac{2\xi}{A} \alpha_1^2 \alpha_2 \alpha_3 \gamma_1 H_0^2 e^{12\alpha_1 H_0^2} \right. \\ & e^{36\alpha_2 H_0^4} + 4 \alpha_1 \alpha_2^2 \alpha_3 H_0^4 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} + \frac{12\xi}{A} \alpha_1 \alpha_2^2 \alpha_3 \gamma_1 H_0^4 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} - \frac{\omega_0 \xi^2}{9A^2} H_0^2 \left. \right) c_2^{\gamma_1} a^{-\xi\gamma_1/3A} \\ & - \gamma_2 c_2^{\gamma_3} a^{-\xi\gamma_3/3A} - \gamma_4 c_2^{\gamma_5} a^{-\xi\gamma_5/3A} + \left( \frac{2\xi c_1}{A} \alpha_1^2 \alpha_2 \alpha_3 \gamma_1 H_0^2 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} - \frac{12\xi c_1}{A} \alpha_1 \alpha_2^2 \alpha_3 \gamma_1 H_0^4 \right. \\ & e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} + \frac{2\omega_0 c_1 \xi^2}{9A^2} H_0^2 \left. \right) c_2^{\gamma_1 - A} a^{-\xi(\gamma_1 - A)/3A} - \frac{\omega_0 \xi^2 c_1^2}{9A^2} H_0^2 c_2^{\gamma_1 - 2A} a^{-\xi(\gamma_1 - 2A)/3A}. \end{aligned} \quad (30)$$

- Case-II:  $\omega(\phi) = \omega_0$

If we choose  $\omega(\phi) = \omega_0$  then we have  $\gamma_1 = 2$ ,  $\gamma_3 = 2$  and we get potential of the form

$$\begin{aligned}
V(a) = & \left( -\alpha_1 \alpha_2 \alpha_3 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} - 18\alpha_1^2 \alpha_2 \alpha_3 H_0^2 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} + \frac{4\xi}{A} \alpha_1^2 \alpha_2 \alpha_3 H_0^2 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} \right. \\
& + 4\alpha_1 \alpha_2^2 \alpha_3 H_0^4 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} + \frac{24\xi}{A} \alpha_1 \alpha_2^2 \alpha_3 H_0^4 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} - \frac{\omega_0 \xi^2}{9A^2} H_0^2 \Big) c_2^2 a^{-2\xi/3A} - \gamma_2 c_2^2 \times \\
& a^{-2\xi/3A} - \gamma_4 c_2^{\gamma_5} a^{-\xi\gamma_5/3A} + \left( \frac{4\xi c_1}{A} \alpha_1^2 \alpha_2 \alpha_3 H_0^2 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} - \frac{24\xi c_1}{A} \alpha_1 \alpha_2^2 \alpha_3 H_0^4 e^{12\alpha_1 H_0^2} e^{36\alpha_2 H_0^4} \right. \\
& \left. + \frac{2\omega_0 c_1 \xi^2}{9A^2} H_0^2 \right) c_2^{2-A} a^{-\xi(2-A)/3A} - \frac{\omega_0 \xi^2 c_1^2}{9A^2} H_0^2 c_2^{2-2A} a^{-\xi(2-2A)/3A}. \quad (31)
\end{aligned}$$

### B. $f(R, \phi)$ Model

Now we are utilizing  $f(R, \phi)$  model, independent of  $Y$  which we have already constructed in the paper [54] and it is given by

$$f(R, \phi) = \alpha_1 \alpha_2 e^{\alpha_1 R} \phi^{\gamma_1} + \gamma_2 \phi^{\gamma_3} + \gamma_4 \phi^{\gamma_5}, \quad (32)$$

where  $\alpha'_i$ 's are constants of integration and

$$\gamma_1 = -\frac{1}{\beta} \left( 1 + \frac{1}{6H_0^2 \alpha_1} \right), \quad \gamma_2 = \omega_0 \beta^2 H_0^2, \quad \gamma_3 = m + 2, \quad \gamma_4 = -2\kappa^2 \rho_0 a_0^{3(1+w)}, \quad \gamma_5 = -\frac{3}{\beta}.$$

- Case-I:  $\omega(\phi) = \omega_0 \phi^m$

Substituting model (32) into (A.1) and choosing  $m = \gamma_1 - 2$ , we have

$$\phi'' + \frac{4}{a} \phi' + \left\{ \gamma_1 - 1 - \frac{\omega_0}{\alpha_1^2 \alpha_2 \gamma_1 e^{\alpha_1 R}} \right\} \frac{\phi'^2}{\phi} = 0. \quad (33)$$

Introducing the variable  $\sigma = \phi'/\phi$  for simplification, we can write the above equation as follows

$$\sigma' + \frac{4}{a} \sigma + \left\{ \gamma_1 - \frac{\omega_0}{\alpha_1^2 \alpha_2 \gamma_1 e^{\alpha_1 R}} \right\} \sigma^2 = 0. \quad (34)$$

With the help of this new function

$$\sigma = \left\{ \gamma_1 - \frac{\omega_0}{\alpha_1^2 \alpha_2 \gamma_1 e^{\alpha_1 R}} \right\}^{-1} \frac{u'}{u} = B^{-1} \frac{u'}{u}, \quad (35)$$

we get the differential equation for  $u$  in the following form

$$u'' + \frac{4}{a} u' = 0. \quad (36)$$

Clearly, we have

$$\phi = u^{1/B}. \quad (37)$$



Equation (8) in case of (32) becomes

$$\begin{aligned} V(\phi) = & -\alpha_1\alpha_2e^{\alpha_1 R}\phi^{\gamma_1} - \gamma_2\phi^{\gamma_3} - \gamma_4\phi^{\gamma_5} - 18\alpha_1^2\alpha_2H_0^2e^{\alpha_1 R}\phi^{\gamma_1} - 6a\alpha_1^2\alpha_2\gamma_1H_0^2\phi'e^{\alpha_1 R}\phi^{\gamma_1-1} \\ & - \omega_0a^2H_0^2\phi^{\gamma_1-2}\phi'^2. \end{aligned} \quad (38)$$

If  $\dot{\phi} \neq 0$ , all the assumptions regarding derivative of (7) are justifiable. The constant field must be discussed separately. First consider, if  $\phi$  is constant, then  $f$  and  $V$  are also independent of time. This leads to the cosmological evolution which is occurred due to cosmological constant. Now we can rewrite Eq. (4) as

$$V = -f - 18H_0^2f_R. \quad (39)$$

Then, on substituting  $\dot{\phi} = 0$  and  $\dot{H} = 0$  into Eq. (6), we obtain

$$\frac{dV}{d\phi} = -\frac{df}{d\phi}. \quad (40)$$

Multiplying Eq. (39) by Eq. (40) and then by integration, we have

$$\begin{aligned} V^2 = & (1 + 18H_0^2\alpha_1)\alpha_1^2\alpha_2^2e^{24\alpha_1 H_0^2}\phi^{2\gamma_1} + \frac{2\alpha_1\alpha_2\gamma_2}{\gamma_1 + \gamma_3}(\gamma_1 + \gamma_3 + 18H_0^2\alpha_1\gamma_3)e^{12\alpha_1 H_0^2}\phi^{\gamma_1+\gamma_3} \\ & + \frac{2\alpha_1\alpha_2\gamma_4}{\gamma_1 + \gamma_5}(\gamma_1 + \gamma_5 + 18H_0^2\alpha_1\gamma_5)e^{12\alpha_1 H_0^2}\phi^{\gamma_1+\gamma_5} + \gamma_2^2\phi^{2\gamma_3} + 2\gamma_2\gamma_4\phi^{\gamma_3+\gamma_5} + \gamma_4^2\phi^{2\gamma_5}. \end{aligned} \quad (41)$$

The basic equation (36) is now given by

$$a^2u'' + 4au' = 0, \quad (42)$$

whose general solution is

$$u(a) = c_1 + c_2a^{-3}, \quad (43)$$

where  $c_1$  and  $c_2$  are integration constants. From (43), we can write

$$a = \left( \frac{c_2}{u - c_1} \right)^{1/3}, \quad (44)$$

and from (37), we have

$$\phi = u^{1/B} = \left( c_1 + \frac{c_2}{a^3} \right)^{1/B}, \quad (45)$$

and inversely, we can write

$$a = \left( \frac{c_2}{\phi^B - c_1} \right)^{1/3}. \quad (46)$$

Further,

$$\phi' a = -\frac{3}{B}\phi(1 - c_1\phi^{-B}). \quad (47)$$

Inserting (45) and (47) into (38), we get

$$\begin{aligned} V(\phi) = & -\alpha_1\alpha_2e^{\alpha_1 R}\phi^{\gamma_1} - \gamma_2\phi^{\gamma_3} - \gamma_4\phi^{\gamma_5} - 18\alpha_1^2\alpha_2H_0^2e^{\alpha_1 R}\phi^{\gamma_1} + \frac{18}{B}\alpha_1^2\alpha_2\gamma_1H_0^2e^{\alpha_1 R}\phi^{\gamma_1} - \frac{9\omega_0}{B^2}H_0^2\phi^{\gamma_1} \\ & - \frac{18c_1}{B}\alpha_1^2\alpha_2\gamma_1H_0^2e^{\alpha_1 R}\phi^{\gamma_1-B} + \frac{18\omega_0c_1}{B^2}H_0^2\phi^{\gamma_1-B} - \frac{9\omega_0c_1^2}{B^2}H_0^2\phi^{\gamma_1-2B}. \end{aligned} \quad (48)$$

If we choose  $c_1 = 0$  in (45), we have  $\phi = c_2a^{-3/B}$  and using it in the above equation, we get the potential in terms of scale factor as follows

$$\begin{aligned} V(\phi) = & -\alpha_1\alpha_2e^{\alpha_1 R}c_2^{\gamma_2}a^{-3\gamma_2/B} - \gamma_2c_2^{\gamma_3}a^{-3\gamma_3/B} - \gamma_4c_2^{\gamma_5}a^{-3\gamma_5/B} - 18\alpha_1^2\alpha_2H_0^2e^{\alpha_1 R}c_2^{\gamma_1}a^{-3\gamma_1/B} + \frac{18}{B} \\ & \alpha_1^2\alpha_2\gamma_1H_0^2e^{\alpha_1 R}c_2^{\gamma_1}a^{-3\gamma_1/B} - \frac{9\omega_0}{B^2}H_0^2c_2^{\gamma_1}a^{-3\gamma_1/B} - \frac{18c_1}{B}\alpha_1^2\alpha_2\gamma_1H_0^2e^{\alpha_1 R}c_2^{\gamma_1-B}a^{-3(\gamma_1-B)/B} + \frac{18\omega_0c_1}{B^2} \\ & H_0^2c_2^{\gamma_1-B}a^{-3(\gamma_1-B)/B} - \frac{9\omega_0c_1^2}{B^2}H_0^2c_2^{\gamma_1-2B}a^{-3(\gamma_1-2B)/B}. \end{aligned} \quad (49)$$

- Case-II:  $\omega(\phi) = \omega_0$

If we choose  $\omega(\phi) = \omega_0$  then we have  $\gamma_1 = 2$ ,  $\gamma_3 = 2$  and we get potential of the form

$$\begin{aligned} V(\phi) = & -\alpha_1\alpha_2e^{\alpha_1 R}c_2^{\gamma_2}a^{-3\gamma_2/B} - \gamma_2c_2^2a^{-6/B} - \gamma_4c_2^{\gamma_5}a^{-3\gamma_5/B} - 18\alpha_1^2\alpha_2H_0^2e^{\alpha_1 R}c_2^2a^{-6/B} + \frac{36}{B}\alpha_1^2\alpha_2 \\ & H_0^2e^{\alpha_1 R}c_2^2a^{-6/B} - \frac{9\omega_0}{B^2}H_0^2c_2^2a^{-6/B} - \frac{36c_1}{B}\alpha_1^2\alpha_2H_0^2e^{\alpha_1 R}c_2^{2-B}a^{-3(2-B)/B} + \frac{18\omega_0c_1}{B^2}H_0^2a^{-3(2-B)/B} \\ & \times c_2^{2-B} - \frac{9\omega_0c_1^2}{B^2}H_0^2c_2^{2-2B}a^{-3(2-2B)/B}. \end{aligned} \quad (50)$$

### C. $f(Y, \phi)$ Model

Here we explore the nature of field potential for  $f(Y, \phi)$  model, independent of  $R$  that is constructed in [54]. It has the following form

$$f(Y, \phi) = \alpha_1\alpha_2e^{\alpha_1 Y}\phi^{\gamma_1} + \gamma_2\phi^{\gamma_3} + \gamma_4\phi^{\gamma_5}, \quad (51)$$

where  $\alpha'_i$ s are constants of integration and

$$\gamma_1 = -\frac{7}{3\beta} + \frac{1}{36\alpha_1\beta H_0^4}, \quad \gamma_2 = \omega_0\beta^2 H_0^2, \quad \gamma_3 = m + 2, \quad \gamma_4 = -2\kappa^2\rho_0 a_0^{3(1+w)}, \quad \gamma_5 = -\frac{3}{\beta}.$$

- Case-I:  $\omega(\phi) = \omega_0\phi^m$

Substituting model (51) into (A.1) and choosing  $m = \gamma_1 - 2$ , we have

$$\phi'' + \frac{8}{9a}\phi' + \left\{ \gamma_1 - 1 - \frac{\omega_0}{6\alpha_1^2\alpha_2\gamma_1 H_0^2 e^{\alpha_1 Y}} \right\} \frac{\phi'^2}{\phi} = 0. \quad (52)$$

Introducing the variable  $\sigma = \phi'/\phi$ , we can write (52) as

$$\sigma' + \frac{8}{9a}\sigma + \left\{ \gamma_1 - \frac{\omega_0}{6\alpha_1^2\alpha_2\gamma_1 H_0^2 e^{\alpha_1 Y}} \right\} \sigma^2 = 0. \quad (53)$$

The introduction of the function

$$\sigma = \frac{\omega_0}{6\alpha_1^2\alpha_2\gamma_1 H_0^2 e^{\alpha_1 Y}}^{-1} \frac{u'}{u} = C^{-1} \frac{u'}{u}, \quad (54)$$

lead to the following differential equation

$$u'' + \frac{8}{9a}u' = 0. \quad (55)$$

On comparing, it is easy to check

$$\phi = u^{1/C}. \quad (56)$$

Equation (8) in case of (51) becomes

$$\begin{aligned} V(\phi) = & -\alpha_1\alpha_2 e^{\alpha_1 Y} \phi^{\gamma_1} - \gamma_2 \phi^{\gamma_3} - \gamma_4 \phi^{\gamma_5} + 4\alpha_1^2\alpha_2 H_0^4 e^{\alpha_1 Y} \phi^{\gamma_1} - 36a\alpha_1^2\alpha_2\gamma_1 H_0^4 \phi' e^{\alpha_1 Y} \phi^{\gamma_1-1} \\ & - \omega_0 a^2 H_0^2 \phi^{\gamma_1-2} \phi'^2. \end{aligned} \quad (57)$$

If  $\dot{\phi} \neq 0$ , all the assumptions regarding derivative of (7) are justifiable. The constant field must be discussed separately. First consider, if  $\phi$  is constant, then  $f$  and  $V$  are also independent of time. This leads to the cosmological evolution which is occurred due to cosmological constant. Now we can rewrite Eq.(4) as

$$V = -f + 4H_0^4 f_Y. \quad (58)$$

Then, on substituting  $\dot{\phi} = 0$  and  $\dot{H} = 0$  into Eq.(6) we have

$$\frac{dV}{d\phi} = -\frac{df}{d\phi}. \quad (59)$$

Multiplying Eq.(58) by Eq.(59) and then by integration, we have

$$\begin{aligned} V^2 = & (1 - 4H_0^4\alpha_1) \alpha_1^2\alpha_2^2 e^{72\alpha_1 H_0^4} \phi^{2\gamma_1} + \frac{2\alpha_1\alpha_2\gamma_2}{\gamma_1 + \gamma_3} (\gamma_1 + \gamma_3 - 4H_0^4\alpha_1\gamma_3) e^{36\alpha_1 H_0^4} \phi^{\gamma_1+\gamma_3} \\ & + \frac{2\alpha_1\alpha_2\gamma_4}{\gamma_1 + \gamma_5} (\gamma_1 + \gamma_5 - 4H_0^4\alpha_1\gamma_5) e^{36\alpha_1 H_0^4} \phi^{\gamma_1+\gamma_5} + \gamma_2^2 \phi^{2\gamma_3} + 2\gamma_2\gamma_4 \phi^{\gamma_3+\gamma_5} + \gamma_4^2 \phi^{2\gamma_5}. \end{aligned} \quad (60)$$

The basic equation (55) is now

$$a^2 u'' + \frac{8}{9} a u' = 0, \quad (61)$$

whose general solution is

$$u(a) = c_1 + c_2 a^{1/9}, \quad (62)$$

where  $c_1$  and  $c_2$  are arbitrary constants. From (62), we can write

$$a = \left( \frac{u - c_1}{c_2} \right)^9, \quad (63)$$

and from (56), we have

$$\phi = u^{1/C} = \left( c_1 + c_2 a^{1/9} \right)^{1/C}, \quad (64)$$

and inversely, it can be written as

$$a = \left( \frac{\phi^C - c_1}{c_2} \right)^9. \quad (65)$$

Furthermore,

$$\phi' a = \frac{1}{9C} \phi (1 - c_1 \phi^{-C}). \quad (66)$$

By using (64) and (66) in Eq.(57), we get

$$\begin{aligned} V(\phi) = & -\alpha_1 \alpha_2 e^{\alpha_1 Y} \phi^{\gamma_1} - \gamma_2 \phi^{\gamma_3} - \gamma_4 \phi^{\gamma_5} + 4\alpha_1^2 \alpha_2 H_0^4 e^{\alpha_1 Y} \phi^{\gamma_1} - \frac{4}{C} \alpha_1^2 \alpha_2 \gamma_1 H_0^4 e^{\alpha_1 Y} \phi^{\gamma_1} - \frac{\omega_0}{81C^2} H_0^2 \phi^{\gamma_1} \\ & + \frac{4c_1}{C} \alpha_1^2 \alpha_2 \gamma_1 H_0^4 e^{\alpha_1 Y} \phi^{\gamma_1 - C} + \frac{2\omega_0 c_1}{81C^2} H_0^2 \phi^{\gamma_1 - C} - \frac{\omega_0 c_1^2}{81C^2} H_0^2 \phi^{\gamma_1 - 2C}. \end{aligned} \quad (67)$$

If we choose  $c_1 = 0$  in (64), we have  $\phi = c_2 a^{1/9C}$  and using it in the above equation, we get the potential in terms of scale factor as follows

$$\begin{aligned} V(a) = & -\alpha_1 \alpha_2 e^{\alpha_1 Y} c_2^{\gamma_1} a^{\gamma_1/9C} - \gamma_2 c_2^{\gamma_3} a^{\gamma_3/9C} - \gamma_4 c_2^{\gamma_5} a^{\gamma_5/9C} + 4\alpha_1^2 \alpha_2 H_0^4 e^{\alpha_1 Y} c_2^{\gamma_1} a^{\gamma_1/9C} - \frac{4}{C} \alpha_1^2 \alpha_2 \\ & \times \gamma_1 H_0^4 e^{\alpha_1 Y} c_2^{\gamma_1} a^{\gamma_1/9C} - \frac{\omega_0}{81C^2} H_0^2 c_2^{\gamma_1} a^{\gamma_1/9C} + \frac{4c_1}{C} \alpha_1^2 \alpha_2 \gamma_1 H_0^4 e^{\alpha_1 Y} c_2^{\gamma_1 - C} a^{\gamma_1 - C/9C} + \frac{2\omega_0 c_1}{81C^2} H_0^2 \\ & \times c_2^{\gamma_1 - C} a^{\gamma_1 - C/9C} - \frac{\omega_0 c_1^2}{81C^2} H_0^2 c_2^{\gamma_1 - 2C} a^{\gamma_1 - 2C/9C}. \end{aligned} \quad (68)$$

- Case-II:  $\omega(\phi) = \omega_0$

If we choose  $\omega(\phi) = \omega_0$  then we have  $\gamma_1 = 2$ ,  $\gamma_3 = 2$  and we get potential of the form

$$\begin{aligned} V(a) = & -\alpha_1 \alpha_2 e^{\alpha_1 Y} c_2^2 a^{2/9C} - \gamma_2 c_2^2 a^{2/9C} - \gamma_4 c_2^{\gamma_5} a^{\gamma_5/9C} + 4\alpha_1^2 \alpha_2 H_0^4 e^{\alpha_1 Y} c_2^2 a^{2/9C} - \frac{8}{C} \alpha_1^2 \alpha_2 H_0^4 e^{\alpha_1 Y} \\ & \times c_2^2 a^{2/9C} - \frac{\omega_0}{81C^2} H_0^2 c_2^2 a^{2/9C} + \frac{8c_1}{C} \alpha_1^2 \alpha_2 H_0^4 e^{\alpha_1 Y} c_2^{2-C} a^{2-C/9C} + \frac{2\omega_0 c_1}{81C^2} H_0^2 c_2^{2-C} a^{2-C/9C} - \frac{\omega_0 c_1^2}{81C^2} \\ & \times H_0^2 c_2^{2-2C} a^{2-2C/9C}. \end{aligned} \quad (69)$$

We cannot discuss barotropic fluid, cosmological constant and chaplygin gas because in de-Sitter universe  $H$  is constant.

#### IV. POWER-LAW MODELS

It would be interesting to study power-law solutions in this modified gravity theory that are indicated by various eras of cosmic evolution. These solutions are helpful to clarify cosmic evolution, with the help of different epochs like dark energy, matter and radiation dominated eras. For this model, scale factor is described as [57, 58]

$$a(t) = a_0 t^n, \quad H(t) = \frac{n}{t}, \quad R = 6n(1 - 2n)t^{-2}, \quad (70)$$

Here we are using [56]

$$\omega(\phi) = \omega_0 \phi^m, \quad \phi(t) \sim a(t)^\beta. \quad (71)$$

##### A. $f(R, \phi)$ Model:

In [54], a well-behaved model  $f(R, \phi)$  has been constructed, here we are interested to evaluate the field potential using this model. The model is defined as

$$f(R, \phi) = \alpha_1 \alpha_2 \phi^{\gamma_1} R^{\gamma_2} + \gamma_3 \phi^{\gamma_4} + \gamma_5 \phi^{\gamma_6}, \quad (72)$$

where  $\alpha'_i$ s are constants of integration and

$$\begin{aligned} \gamma_1 &= \frac{\alpha_1}{3n-1} + \frac{n-3}{n\beta} - \frac{2(3n-1)^2}{n^2\beta^2\alpha_1}, \quad \gamma_2 = \frac{n(n-3)\beta\alpha_1}{(3n-1)^2}, \quad \gamma_3 = \omega_0\beta^2 n^2 a_0^{\frac{2}{n}}, \\ \gamma_4 &= m+2 - \frac{2}{n\beta}, \quad \gamma_5 = -2\kappa^2 \rho_0 a_0^{3(1+w)}, \quad \gamma_6 = -\frac{3}{\beta}. \end{aligned}$$

Substituting model (72) into (A.1) and choosing  $(m - \gamma_1 + 1 - \frac{2}{n\beta} + \frac{2\gamma_2}{n\beta}) = -1$ , we have

$$\begin{aligned} \phi'' + \frac{1}{a} \left\{ \frac{4n-3}{n} - \frac{2(\gamma_2-1)}{n} \right\} \phi' + \left\{ \gamma_1 - 1 - \frac{\omega_0(6n-12n^2)^{1-\gamma_2} a_0^{(2-2\gamma_2)/n}}{\alpha_1 \alpha_2 \gamma_1 \gamma_2} \right\} \frac{\phi'^2}{\phi} \\ + \frac{1}{a^2} \left\{ \frac{2-6n}{n^2\gamma_1} + \frac{8(\gamma_2-1)}{n^2\gamma_1} \right\} \phi = 0. \end{aligned} \quad (73)$$

Introducing the variable  $\sigma = \phi'/\phi$ , we can write (73) as

$$\begin{aligned} \sigma' + \frac{1}{a} \left\{ \frac{4n-3}{n} - \frac{2(\gamma_2-1)}{n} \right\} \sigma + \left\{ \gamma_1 - \frac{\omega_0(6n-12n^2)^{1-\gamma_2} a_0^{(2-2\gamma_2)/n}}{\alpha_1 \alpha_2 \gamma_1 \gamma_2} \right\} \sigma^2 \\ + \frac{1}{a^2} \left\{ \frac{2-6n}{n^2\gamma_1} + \frac{8(\gamma_2-1)}{n^2\gamma_1} \right\} = 0. \end{aligned} \quad (74)$$

On introducing a new function

$$\sigma = \left\{ \gamma_1 - \frac{\omega_0(6n - 12n^2)^{1-\gamma_2} a_0^{(2-2\gamma_2)/n}}{\alpha_1 \alpha_2 \gamma_1 \gamma_2} \right\}^{-1} \frac{u'}{u} = D^{-1} \frac{u'}{u}, \quad (75)$$

we get differential equation of the form

$$u'' + \frac{\xi}{a} u' + \frac{D\eta}{a^2} u = 0, \quad (76)$$

where

$$D = \left\{ \gamma_1 - \frac{\omega_0(6n - 12n^2)^{1-\gamma_2} a_0^{(2-2\gamma_2)/n}}{\alpha_1 \alpha_2 \gamma_1 \gamma_2} \right\}, \quad \eta = \left\{ \frac{2-6n}{n^2 \gamma_1} + \frac{8(\gamma_2-1)}{n^2 \gamma_1} \right\} \\ \xi = \left\{ \frac{4n-3}{n} - \frac{2(\gamma_2-1)}{n} \right\}. \quad (77)$$

On comparison, we have

$$\phi = u^{1/D}. \quad (78)$$

Eq.(8) in case of (72) becomes

$$V(\phi) = -\alpha_1 \alpha_2 \phi^{\gamma_1} R^{\gamma_2} - \gamma_3 \phi^{\gamma_4} - \gamma_5 \phi^{\gamma_6} - 6\alpha_1 \alpha_2 \gamma_2 (3H^2 + aHH') \phi^{\gamma_1} R^{\gamma_2-1} + 36\alpha_1 \alpha_2 \gamma_2 (\gamma_2 - 1) \\ \times H \left( a^2 H^2 H'' + a^2 HH'^2 + 5aH^2 H' \right) \phi^{\gamma_1} R^{\gamma_2-2} - 6a\alpha_1 \alpha_2 \gamma_1 \gamma_2 H^2 \phi' \phi^{\gamma_1-1} R^{\gamma_2-1} \\ - \omega_0 a^2 H^2 \phi^m \phi'^2. \quad (79)$$

If  $\dot{\phi} \neq 0$ , all the assumptions regarding derivative of (7) are justifiable. The constant field must be discussed separately. First consider, if  $\phi$  is constant, then  $f$  and  $V$  are also independent of time and from Friedmann equation (4), it can be noticed that the Hubble parameter  $H$  is also a constant. This leads to the cosmological evolution which is occurred due to cosmological constant. Now we can rewrite Eq.(4) as follows

$$V = -f - 18H^2 f_R. \quad (80)$$

Then, on substituting  $\dot{\phi} = 0$  and  $\dot{H} = 0$  into Eq.(6), we have

$$\frac{dV}{d\phi} = -\frac{df}{d\phi}. \quad (81)$$

Multiplying equations (80), (81) and substituting model (72), and then by integrating, we have the potential in this form

$$V^2 = \frac{n\beta\alpha_1^2\alpha_2^2\gamma_1}{n\beta\gamma_1 - 2\gamma_2} (R + 18H^2\gamma_2) R^{2\gamma_2-1} \phi^{2\gamma_1} + \frac{2n\beta\alpha_1\alpha_2\gamma_3}{(\gamma_1 + \gamma_4)n\beta - 2\gamma_2} (\gamma_1 R + \gamma_4 R + 18H^2\gamma_2\gamma_4) R^{\gamma_2-1} \\ \times \phi^{\gamma_1+\gamma_4} + \frac{2n\beta\alpha_1\alpha_2\gamma_5}{(\gamma_1 + \gamma_6)n\beta - 2\gamma_2} (\gamma_1 R + \gamma_6 R + 18H^2\gamma_2\gamma_6) R^{\gamma_2-1} \phi^{\gamma_1+\gamma_6} + \gamma_3^2 \phi^{2\gamma_4} + 2\gamma_3\gamma_5 \phi^{\gamma_4+\gamma_6} \\ + \gamma_5^2 \phi^{2\gamma_6} \quad (82)$$

### 1. Barotropic Fluid

Equation of state parameter for barotropic fluid is

$$p = w\rho, \quad (83)$$

and Hubble parameter defined as

$$H(a) = H_0 a^{-\frac{3}{2}(1+w)}. \quad (84)$$

The basic equation (76) is now

$$a^2 u'' + a\xi u' + D\eta u = 0, \quad (85)$$

whose general solution is

$$u(a) = c_1 a^{p_1} + c_2 a^{p_2}, \quad (86)$$

where  $c_1$  and  $c_2$  are integration constants, and the exponents  $p_1$  and  $p_2$  are

$$p_{1,2} = \left( \frac{1-\xi}{2} \right) \pm \sqrt{\left( \frac{1-\xi}{2} \right)^2 - D\eta}. \quad (87)$$

For the sake of simplicity, we choose one of the constants  $c_1$  or  $c_2$  as zero. In this situation, one can hope to modify the function (86). For this choice, we have

$$a = u^{1/p_{1,2}}, \quad (88)$$

and from (78), we have

$$\phi = u^{1/D} = a^{\frac{p_{1,2}}{D}}, \quad (89)$$

and inversely, we can write

$$a = \phi^{D/p_{1,2}}. \quad (90)$$

Furthermore,

$$\phi' a = \frac{p_{1,2}}{D} \phi, \quad (91)$$

- Case-I:  $\omega(\phi) = \omega_0 \phi^m$

Inserting (84), (89) and (91) into Eq.(79), we have

$$\begin{aligned}
V_{1,2} = & -\alpha_1\alpha_2 3^{\gamma_2} H_0^{2\gamma_2} (3w-1)^{\gamma_2} \phi^{\gamma_1 - \frac{3(1+w)\gamma_2 D}{p_{1,2}}} - \gamma_3 \phi^{\gamma_4} - \gamma_5 \phi^{\gamma_6} - 3^{\gamma_2+1} \alpha_1 \alpha_2 \gamma_2 H_0^{2\gamma_2} (3w-1)^{\gamma_2-1} \\
& \times (1-w) \phi^{\gamma_1 - \frac{3(1+w)\gamma_2 D}{p_{1,2}}} + 18\alpha_1 \alpha_2 \gamma_2 (\gamma_2 - 1) 3^{\gamma_2} (1+w)^2 (3w-1)^{\gamma_2-2} H_0^{2\gamma_2} \phi^{\gamma_1 - \frac{3(1+w)\gamma_2 D}{p_{1,2}}} \\
& - \frac{2p_{1,2}}{D} \alpha_1 \alpha_2 \gamma_1 \gamma_2 3^{\gamma_2} H_0^{2\gamma_2} (3w-1)^{\gamma_2-1} \phi^{\gamma_1 - \frac{3(1+w)\gamma_2 D}{p_{1,2}}} - \frac{\omega_0 p_{1,2}^2}{D^2} H_0^2 \phi^{m+2 - \frac{3(1+w)D}{p_{1,2}}}. \tag{92}
\end{aligned}$$

Using (89) in the above equation, we get the scalar potential in terms of scale factor as follows

$$\begin{aligned}
V_{1,2} = & -\alpha_1\alpha_2 3^{\gamma_2} H_0^{2\gamma_2} (3w-1)^{\gamma_2} a^{\xi_1} - \gamma_3 \phi^{\gamma_4} - \gamma_5 \phi^{\gamma_6} - 3^{\gamma_2+1} \alpha_1 \alpha_2 \gamma_2 H_0^{2\gamma_2} (3w-1)^{\gamma_2-1} (1-w) a^{\xi_1} \\
& + 18\alpha_1 \alpha_2 \gamma_2 (\gamma_2 - 1) 3^{\gamma_2} (1+w)^2 (3w-1)^{\gamma_2-2} H_0^{2\gamma_2} a^{\xi_1} - \frac{2p_{1,2}}{D} \alpha_1 \alpha_2 \gamma_1 \gamma_2 3^{\gamma_2} H_0^{2\gamma_2} (3w-1)^{\gamma_2-1} a^{\xi_1} \\
& - \frac{\omega_0 p_{1,2}^2}{D^2} H_0^2 a^{\xi_2}, \tag{93}
\end{aligned}$$

where  $\xi_1 = -3(1+w)\gamma_2 + \frac{\gamma_1 p_{1,2}}{D}$  and  $\xi_2 = -3(1+w) + \frac{(m+2)p_{1,2}}{D}$ .

- Case-II:  $\omega(\phi) = \omega_0$

If we choose  $\omega(\phi) = \omega_0$  then we have  $\gamma_1 + \frac{2}{n\beta} - \frac{2\gamma_2}{n\beta} = 2$ ,  $\gamma_4 = 2 - \frac{2}{n\beta}$  and potential is of the form

$$\begin{aligned}
V_{1,2} = & -\alpha_1\alpha_2 3^{\gamma_2} H_0^{2\gamma_2} (3w-1)^{\gamma_2} a^{\xi_1} - \gamma_3 \phi^{\gamma_4} - \gamma_5 \phi^{\gamma_6} - 3^{\gamma_2+1} \alpha_1 \alpha_2 \gamma_2 H_0^{2\gamma_2} (3w-1)^{\gamma_2-1} (1-w) a^{\xi_1} \\
& + 18\alpha_1 \alpha_2 \gamma_2 (\gamma_2 - 1) 3^{\gamma_2} (1+w)^2 (3w-1)^{\gamma_2-2} H_0^{2\gamma_2} a^{\xi_1} - \frac{2p_{1,2}}{D} \alpha_1 \alpha_2 \gamma_1 \gamma_2 3^{\gamma_2} H_0^{2\gamma_2} (3w-1)^{\gamma_2-1} a^{\xi_1} \\
& - \frac{\omega_0 p_{1,2}^2}{D^2} H_0^2 a^{\xi_2}. \tag{94}
\end{aligned}$$

where  $\xi_1 = -3(1+w)\gamma_2 + \frac{\gamma_1 p_{1,2}}{D}$  and  $\xi_2 = -3(1+w) + \frac{2p_{1,2}}{D}$ .

## 2. Cosmological Constant

- Case-I:  $\omega(\phi) = \omega_0 \phi^m$

If  $w = -1$  then we have

$$H = H_0, \quad R = 12H_0^2$$

and consequently Eq.(92) transforms to

$$\begin{aligned}
V_{1,2}(\phi) = & -\alpha_1\alpha_2 (12)^{\gamma_2} H_0^{2\gamma_2} \phi^{\gamma_1} - \gamma_3 \phi^{\gamma_4} - \gamma_5 \phi^{\gamma_6} - 18\alpha_1 \alpha_2 \gamma_2 (12)^{\gamma_2-1} H_0^{2\gamma_2} \phi^{\gamma_1} - \frac{p_{1,2}}{2D} \alpha_1 \alpha_2 \gamma_1 \gamma_2 \\
& \times (12)^{\gamma_2} H_0^{2\gamma_2} \phi^{\gamma_1} - \frac{\omega_0 p_{1,2}^2}{D^2} H_0^2 \phi^{m+2}. \tag{95}
\end{aligned}$$



In terms of scale factor, the above equation can be written as

$$V_{1,2}(a) = -\alpha_1\alpha_2(12)^{\gamma_2}H_0^{2\gamma_2}a^{\frac{\gamma_1 p_{1,2}}{D}} - \gamma_3 a^{\frac{\gamma_4 p_{1,2}}{D}} - \gamma_5 a^{\frac{\gamma_6 p_{1,2}}{D}} - 18\alpha_1\alpha_2\gamma_2(12)^{\gamma_2-1}H_0^{2\gamma_2}a^{\frac{\gamma_1 p_{1,2}}{D}} - \frac{p_{1,2}}{2D} \\ \times \alpha_1\alpha_2\gamma_1\gamma_2(12)^{\gamma_2}H_0^{2\gamma_2}a^{\frac{\gamma_1 p_{1,2}}{D}} - \frac{\omega_0 p_{1,2}^2}{D^2}H_0^2 a^{(m+2)p_{1,2}/D}. \quad (96)$$

- Case-II:  $\omega(\phi) = \omega_0$

For constant coupling, we have  $m = 0$  then  $\gamma_1 + \frac{2}{n\beta} - \frac{2\gamma_2}{n\beta} = 2$ ,  $\gamma_4 = 2 - \frac{2}{n\beta}$  and then potential takes the form

$$V_{1,2}(a) = -\alpha_1\alpha_2(12)^{\gamma_2}H_0^{2\gamma_2}a^{\frac{\gamma_1 p_{1,2}}{D}} - \gamma_3 a^{\frac{\gamma_4 p_{1,2}}{D}} - \gamma_5 a^{\frac{\gamma_6 p_{1,2}}{D}} - 18\alpha_1\alpha_2\gamma_2(12)^{\gamma_2-1}H_0^{2\gamma_2}a^{\frac{\gamma_1 p_{1,2}}{D}} - \frac{p_{1,2}}{2D} \\ \times \alpha_1\alpha_2\gamma_1\gamma_2(12)^{\gamma_2}H_0^{2\gamma_2}a^{\frac{\gamma_1 p_{1,2}}{D}} - \frac{\omega_0 p_{1,2}^2}{D^2}H_0^2 a^{2p_{1,2}/D}. \quad (97)$$

### 3. Chaplygin Gas

The Chaplygin gas model [50] is one of the basic model of dark energy and dark matter that has gained a certain recognition [50, 59, 60]. For Chaplygin gas, the equation of state is

$$p = \frac{\hat{A}}{\rho}, \quad (98)$$

with Hubble parameter of the form

$$H(a) = \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1/4}, \quad (99)$$

where  $\hat{A}$  is a constant.

- Case-I:  $\omega(\phi) = \omega_0\phi^m$

Using Eq.(99), the equation (79) takes the form

$$V_{1,2} = -\alpha_1\alpha_2(-6)^{\gamma_2} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-\frac{1}{2}\gamma_2} \left( 2\hat{A} + \frac{\hat{B}}{2a^6} \right)^{\gamma_2} \phi^{\gamma_1} - \gamma_3 \phi^{\gamma_4} - \gamma_5 \phi^{\gamma_6} + \alpha_1\alpha_2\gamma_2(-6)^{\gamma_2} \\ \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-\frac{1}{2}\gamma_2} \left( 2\hat{A} + \frac{\hat{B}}{2a^6} \right)^{\gamma_2-1} \left( 3\hat{A} + \frac{3\hat{B}}{2a^6} \right) \phi^{\gamma_1} + \alpha_1\alpha_2\gamma_2(\gamma_2-1)(-6)^{\gamma_2} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-\frac{1}{2}\gamma_2} \\ \left( 2\hat{A} + \frac{\hat{B}}{2a^6} \right)^{\gamma_2-2} \left( \frac{3\hat{A}\hat{B}}{a^6} - \frac{3\hat{B}^2}{2a^{12}} \right) \phi^{\gamma_1} + \frac{p_{1,2}}{D} \alpha_1\alpha_2\gamma_1\gamma_2(-6)^{\gamma_2} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1-\frac{1}{2}\gamma_2} \left( 2\hat{A} + \frac{\hat{B}}{2a^6} \right)^{\gamma_2-1} \\ - \frac{\omega_0 p_{1,2}^2}{D^2} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1/2} \phi^{m+2}. \quad (100)$$

In terms of scale factor, the potential for Chaplygin gas is defined by

$$\begin{aligned}
V_{1,2}(a) = & -\alpha_1\alpha_2(-6)^{\gamma_2} \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{-\frac{1}{2}\gamma_2} \left(2\hat{A} + \frac{\hat{B}}{2a^6}\right)^{\gamma_2} a^{\frac{\gamma_1 p_{1,2}}{D}} - \gamma_3 a^{\frac{\gamma_4 p_{1,2}}{D}} - \gamma_5 a^{\frac{\gamma_6 p_{1,2}}{D}} + \alpha_1\alpha_2\gamma_2 \\
& (-6)^{\gamma_2} \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{-\frac{1}{2}\gamma_2} \left(2\hat{A} + \frac{\hat{B}}{2a^6}\right)^{\gamma_2-1} \left(3\hat{A} + \frac{3\hat{B}}{2a^6}\right) a^{\frac{\gamma_1 p_{1,2}}{D}} + \alpha_1\alpha_2\gamma_2(\gamma_2-1) \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{-\frac{1}{2}\gamma_2} \\
& (-6)^{\gamma_2} \left(2\hat{A} + \frac{\hat{B}}{2a^6}\right)^{\gamma_2-2} \left(\frac{3\hat{A}\hat{B}}{a^6} - \frac{3\hat{B}^2}{2a^{12}}\right) a^{\frac{\gamma_1 p_{1,2}}{D}} + \frac{p_{1,2}}{D} \alpha_1\alpha_2\gamma_1\gamma_2(-6)^{\gamma_2} \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{1-\frac{1}{2}\gamma_2} \\
& \left(2\hat{A} + \frac{\hat{B}}{2a^6}\right)^{\gamma_2-1} - \frac{\omega_0 p_{1,2}^2}{D^2} \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{1/2} a^{\frac{(m+2)p_{1,2}}{D}}. \tag{101}
\end{aligned}$$

- Case-II:  $\omega(\phi) = \omega_0$

For  $m = 0$ , i.e., constant coupling, we have  $\gamma_1 + \frac{2}{n\beta} - \frac{2\gamma_2}{n\beta} = 2$ ,  $\gamma_4 = 2 - \frac{2}{n\beta}$  and thus potential of the following form

$$\begin{aligned}
V_{1,2}(a) = & -\alpha_1\alpha_2(-6)^{\gamma_2} \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{-\frac{1}{2}\gamma_2} \left(2\hat{A} + \frac{\hat{B}}{2a^6}\right)^{\gamma_2} a^{\frac{\gamma_1 p_{1,2}}{D}} - \gamma_3 a^{\frac{\gamma_4 p_{1,2}}{D}} - \gamma_5 a^{\frac{\gamma_6 p_{1,2}}{D}} + \alpha_1\alpha_2\gamma_2 \\
& (-6)^{\gamma_2} \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{-\frac{1}{2}\gamma_2} \left(2\hat{A} + \frac{\hat{B}}{2a^6}\right)^{\gamma_2-1} \left(3\hat{A} + \frac{3\hat{B}}{2a^6}\right) a^{\frac{\gamma_1 p_{1,2}}{D}} + \alpha_1\alpha_2\gamma_2(\gamma_2-1) \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{-\frac{1}{2}\gamma_2} \\
& (-6)^{\gamma_2} \left(2\hat{A} + \frac{\hat{B}}{2a^6}\right)^{\gamma_2-2} \left(\frac{3\hat{A}\hat{B}}{a^6} - \frac{3\hat{B}^2}{2a^{12}}\right) a^{\frac{\gamma_1 p_{1,2}}{D}} + \frac{p_{1,2}}{D} \alpha_1\alpha_2\gamma_1\gamma_2(-6)^{\gamma_2} \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{1-\frac{1}{2}\gamma_2} \\
& \left(2\hat{A} + \frac{\hat{B}}{2a^6}\right)^{\gamma_2-1} - \frac{\omega_0 p_{1,2}^2}{D^2} \left(\hat{A} + \frac{\hat{B}}{a^6}\right)^{1/2} a^{\frac{2p_{1,2}}{D}}. \tag{102}
\end{aligned}$$

### B. $f(Y, \phi)$ Model

Here we use  $f(Y, \phi)$  model that is constructed in a recent paper [54]. The model is described as

$$f(Y, \phi) = \alpha_1\alpha_2\phi^{\gamma_1}Y^{\gamma_2} + \gamma_3\phi^{\gamma_4} + \gamma_5\phi^{\gamma_6}, \tag{103}$$

where  $\alpha'_i$ 's are constants of integration and

$$\begin{aligned}
\gamma_1 &= \frac{2(3n-2)\alpha_1}{4n^2-3n+1} + \frac{7n^2-31n+12}{n(3n-2)} - \frac{2(4n^2-3n+1)^2}{n^2(3n-2)^2\alpha_1}, \quad \gamma_2 = \frac{n(3n-2)\alpha_1}{2(4n^2-3n+1)}, \\
\gamma_3 &= -\omega_0\beta^2n^2a_0^{\frac{2}{n}}, \quad \gamma_4 = m+2 - \frac{2}{n\beta}, \quad \gamma_5 = -2\kappa^2\rho_0a_0^{3(1+w)}, \quad \gamma_6 = -\frac{3}{\beta}.
\end{aligned}$$

Substituting model (103) into (A.1) and choosing  $(m - \gamma_1 + 1 - \frac{2}{n\beta} + \frac{4\gamma_2}{n\beta}) = -1$ , we have

$$\begin{aligned} \phi'' + \frac{1}{a} \left\{ \frac{32n^3 - 180n^2 + 112n - 12}{12n^2(3n-2)} - \frac{(\gamma_2 - 1)(72n^3 - 144n^2 + 100n - 24)}{n(3n-2)(6n^2 - 8n + 3)} \right\} \phi' \\ + \left\{ \gamma_1 - 1 - \frac{\omega_0(6)^{1-\gamma_2} n^{1-2\gamma_2} a_0^{(2-4\gamma_2)/n} (6n^2 - 8n + 3)^{1-\gamma_2}}{2(3n-2)\alpha_1\alpha_2\gamma_1\gamma_2} \right\} \frac{\phi'^2}{\phi} + \frac{1}{a^2} \left\{ \frac{24 - 32n + 24n^2 + 8n^3}{6n^3\gamma_1(3n-2)} \right. \\ \left. + \frac{(\gamma_2 - 1)(610n^3 - 1152n^2 + 800n - 192)}{n^2\gamma_1(3n-2)(6n^2 - 8n + 3)} \right\} \phi = 0. \end{aligned} \quad (104)$$

Introducing the variable  $\sigma = \phi'/\phi$ , we can write (104) as

$$\begin{aligned} \sigma' + \frac{1}{a} \left\{ \frac{32n^3 - 180n^2 + 112n - 12}{12n^2(3n-2)} - \frac{(\gamma_2 - 1)(72n^3 - 144n^2 + 100n - 24)}{n(3n-2)(6n^2 - 8n + 3)} \right\} \sigma \\ + \left\{ \gamma_1 - \frac{\omega_0(6)^{1-\gamma_2} n^{1-2\gamma_2} a_0^{(2-4\gamma_2)/n} (6n^2 - 8n + 3)^{1-\gamma_2}}{2(3n-2)\alpha_1\alpha_2\gamma_1\gamma_2} \right\} \sigma^2 + \frac{1}{a^2} \left\{ \frac{24 - 32n + 24n^2 + 8n^3}{6n^3\gamma_1(3n-2)} \right. \\ \left. + \frac{(\gamma_2 - 1)(610n^3 - 1152n^2 + 800n - 192)}{n^2\gamma_1(3n-2)(6n^2 - 8n + 3)} \right\} \phi = 0. \end{aligned} \quad (105)$$

On introducing a new function

$$\sigma = \left\{ \gamma_1 - \frac{\omega_0(6)^{1-\gamma_2} n^{1-2\gamma_2} a_0^{(2-4\gamma_2)/n} (6n^2 - 8n + 3)^{1-\gamma_2}}{2(3n-2)\alpha_1\alpha_2\gamma_1\gamma_2} \right\}^{-1} \frac{u'}{u} = E^{-1} \frac{u'}{u}, \quad (106)$$

we get differential equation of the form

$$u'' + \frac{\xi}{a} u' + \frac{E\eta}{a^2} u = 0, \quad (107)$$

where

$$\begin{aligned} E &= \left\{ \gamma_1 - \frac{\omega_0(6)^{1-\gamma_2} n^{1-2\gamma_2} a_0^{(2-4\gamma_2)/n} (6n^2 - 8n + 3)^{1-\gamma_2}}{2(3n-2)\alpha_1\alpha_2\gamma_1\gamma_2} \right\}, \\ \xi &= \left\{ \frac{32n^3 - 180n^2 + 112n - 12}{12n^2(3n-2)} - \frac{(\gamma_2 - 1)(72n^3 - 144n^2 + 100n - 24)}{n(3n-2)(6n^2 - 8n + 3)} \right\} \\ \eta &= \left\{ \frac{24 - 32n + 24n^2 + 8n^3}{6n^3\gamma_1(3n-2)} + \frac{(\gamma_2 - 1)(610n^3 - 1152n^2 + 800n - 192)}{n^2\gamma_1(3n-2)(6n^2 - 8n + 3)} \right\}. \end{aligned} \quad (108)$$

It is easy to see, on comparing that

$$\phi = u^{1/E}. \quad (109)$$

Equation (8) in case of (103) becomes

$$\begin{aligned} V(\phi) = -\alpha_1\alpha_2\phi^{\gamma_1}Y^{\gamma_2} - \gamma_3\phi^{\gamma_4} - \gamma_5\phi^{\gamma_6} - 2 \left( a^3H^3H''' + 4a^3H^2H'H'' + 7a^2H^3H'' + 8a^2H^2H'^2 \right. \\ \left. + a^3HH'^3 + 11aH^3H' - 2H^4 \right) \alpha_1\alpha_2\gamma_2\phi^{\gamma_1}Y^{\gamma_2-1} - 72H \left( 24a^2H^6H'' + 34a^3H^5H'H'' + 96aH^6H' \right. \\ \left. + 154a^2H^5H'^2 + 78a^3H^4H'^3 + 12a^4H^3H'^4 + 12a^4H^4H'^2H'' \right) \alpha_1\alpha_2\gamma_2(\gamma_2 - 1)\phi^{\gamma_1}Y^{\gamma_2-2} - 12aH^2 \\ \times \phi' \left( 2aHH' + 3H^2 \right) \alpha_1\alpha_2\gamma_1\gamma_2\phi^{\gamma_1}Y^{\gamma_2-1} - \omega_0a^2H^2\phi^m\phi'^2. \end{aligned} \quad (110)$$

If  $\dot{\phi} \neq 0$ , all the assumptions regarding derivative of (7) are justifiable. The constant field must be discussed separately. First consider, if  $\phi$  is constant, then  $f$  and  $V$  are also independent of time and from Friedmann equation (4), it can be noticed that the Hubble parameter  $H$  is also a constant. This leads to the cosmological evolution which is occurred due to cosmological constant. Now we can rewrite Eq.(4) as

$$V = -f + 4H^4 f_Y. \quad (111)$$

Then, on substituting  $\dot{\phi} = 0$  and  $\dot{H} = 0$  into Eq. (6), we have

$$\frac{dV}{d\phi} = -\frac{df}{d\phi}. \quad (112)$$

Multiplying equations (111), (112) and substituting model (103), the scalar potential take the following form

$$\begin{aligned} V^2 = & \frac{n\beta\alpha_1^2\alpha_2^2\gamma_1}{n\beta\gamma_1 - 4\gamma_2} (Y - 4H^4\gamma_2) Y^{2\gamma_2-1} \phi^{2\gamma_1} + \frac{2n\beta\alpha_1\alpha_2\gamma_3}{(\gamma_1 + \gamma_4)n\beta - 4\gamma_2} (\gamma_4 Y + \gamma_1 Y - 4H^4\gamma_2\gamma_4) Y^{\gamma_2-1} \\ & \times \phi^{\gamma_1+\gamma_4} + \frac{2n\beta\alpha_1\alpha_2\gamma_5}{(\gamma_1 + \gamma_6)n\beta - 4\gamma_2} (\gamma_1 Y + \gamma_6 Y - 4H^4\gamma_2\gamma_6) Y^{\gamma_2-1} \phi^{\gamma_1+\gamma_6} + \gamma_3^2 \phi^{2\gamma_4} + 2\gamma_3\gamma_5 \phi^{\gamma_4+\gamma_6} \\ & + \gamma_5^2 \phi^{2\gamma_6} \end{aligned} \quad (113)$$

### 1. Barotropic Fluid

Equation of state parameter for barotropic fluid is

$$p = w\rho, \quad (114)$$

and Hubble parameter defined as

$$H(a) = H_0 a^{-\frac{3}{2}(1+w)}, \quad (115)$$

- Case-I:  $\omega(\phi) = \omega_0 \phi^m$

The basic equation (107) is now

$$a^2 u'' + a\xi u' + E\eta u = 0, \quad (116)$$

whose general solution is

$$u(a) = c_1 a^{p_1} + c_2 a^{p_2}, \quad (117)$$

where  $c_1$  and  $c_2$  are arbitrary constants, and the exponents  $p_1$  and  $p_2$  are

$$p_{1,2} = \left( \frac{1-\xi}{2} \right) \pm \sqrt{\left( \frac{1-\xi}{2} \right)^2 - E\eta}. \quad (118)$$

We shall always choose one of the constants  $c_1$  or  $c_2$  as zero. Just in this situation, one can hope to modify the function (117). At that point, up to a constant

$$a = u^{1/p_{1,2}}, \quad (119)$$

and from (109), we have

$$\phi = u^{1/E} = a^{p_{1,2}/E}, \quad (120)$$

and inversely, we have

$$a = \phi^{E/p_{1,2}}, \quad (121)$$

which further leads to

$$\phi' a = \frac{p_{1,2}}{E} \phi. \quad (122)$$

Utilizing (115), (120) and (122) into (110), we have

$$\begin{aligned} V_{1,2}(\phi) = & -\alpha_1 \alpha_2 (9/2)^{\gamma_2} H_0^{4\gamma_2} (9w^2 + 2w + 1)^{\gamma_2} \phi^{\gamma_1 - \frac{6(1+w)\gamma_2 E}{p_{1,2}}} - \gamma_3 \phi^{\gamma_4} - \gamma_5 \phi^{\gamma_6} + \frac{1}{4} \alpha_1 \alpha_2 \gamma_2 (9/2)^{\gamma_2-1} \\ & \times H_0^{4\gamma_2} (106 + 270w + 342w^2 + 162w^3) (9w^2 + 2w + 1)^{\gamma_2-1} \phi^{\gamma_1 - \frac{6(1+w)\gamma_2 E}{p_{1,2}}} - 18\alpha_1 \alpha_2 \gamma_2 (\gamma_2 - 1) \\ & \times (9/2)^{\gamma_2-2} H_0^{4\gamma_2} (486w^4 + 594w^3 + 162w^2 - 498w - 552) (9w^2 + 2w + 1)^{\gamma_2-2} \phi^{\gamma_1 - \frac{6(1+w)\gamma_2 E}{p_{1,2}}} \\ & + \frac{36wp_{1,2}}{E} \alpha_1 \alpha_2 \gamma_1 \gamma_2 (9/2)^{\gamma_2-1} H_0^{4\gamma_2} (9w^2 + 2w + 1)^{\gamma_2-1} \phi^{\gamma_1 - \frac{6(1+w)\gamma_2 E}{p_{1,2}}} - \frac{\omega_0 H_0^2 p_{1,2}^2}{E^2} \phi^{m+2 - \frac{3(1+w)E}{p_{1,2}}}. \end{aligned} \quad (123)$$

In terms of scale factor, we have the scalar potential as

$$\begin{aligned} V_{1,2}(a) = & -\alpha_1 \alpha_2 (9/2)^{\gamma_2} H_0^{4\gamma_2} (9w^2 + 2w + 1)^{\gamma_2} a^{\xi_1} - \gamma_3 a^{\gamma_4} - \gamma_5 a^{\gamma_6} + \frac{1}{4} \alpha_1 \alpha_2 \gamma_2 (9/2)^{\gamma_2-1} H_0^{4\gamma_2} (106 \\ & + 270w + 342w^2 + 162w^3) (9w^2 + 2w + 1)^{\gamma_2-1} a^{\xi_1} - 18\alpha_1 \alpha_2 \gamma_2 (\gamma_2 - 1) (9/2)^{\gamma_2-2} H_0^{4\gamma_2} (486w^4 \\ & + 594w^3 + 162w^2 - 498w - 552) (9w^2 + 2w + 1)^{\gamma_2-2} a^{\xi_1} + \frac{36wp_{1,2}}{E} \alpha_1 \alpha_2 \gamma_1 \gamma_2 (9/2)^{\gamma_2-1} H_0^{4\gamma_2} a^{\xi_1} \\ & (9w^2 + 2w + 1)^{\gamma_2-1} - \frac{\omega_0 H_0^2 p_{1,2}^2}{E^2} a^{\xi_2}, \end{aligned} \quad (124)$$

where  $\xi_1 = -6(1+w)\gamma_2 + \frac{\gamma_1 p_{1,2}}{E}$  and  $\xi_2 = -3(1+w) + \frac{(m+2)p_{1,2}}{E}$ .

- Case-II:  $\omega(\phi) = \omega_0$

For  $\omega(\phi) = \omega_0$ , we have  $\gamma_1 + \frac{2}{n\beta} - \frac{4\gamma_2}{n\beta} = 2$ ,  $\gamma_4 = 2 - \frac{2}{n\beta}$  and potential of the form

$$\begin{aligned} V_{1,2}(a) = & -\alpha_1\alpha_2(9/2)^{\gamma_2}H_0^{4\gamma_2}(9w^2+2w+1)^{\gamma_2}a^{\xi_1} - \gamma_3a^{\gamma_4} - \gamma_5a^{\gamma_6} + \frac{1}{4}\alpha_1\alpha_2\gamma_2(9/2)^{\gamma_2-1}H_0^{4\gamma_2}(106 \\ & + 270w + 342w^2 + 162w^3)(9w^2+2w+1)^{\gamma_2-1}a^{\xi_1} - 18\alpha_1\alpha_2\gamma_2(\gamma_2-1)(9/2)^{\gamma_2-2}H_0^{4\gamma_2}(486w^4 \\ & + 594w^3 + 162w^2 - 498w - 552)(9w^2+2w+1)^{\gamma_2-2}a^{\xi_1} + \frac{36wp_{1,2}}{E}\alpha_1\alpha_2\gamma_1\gamma_2(9/2)^{\gamma_2-1}H_0^{4\gamma_2}a^{\xi_1} \\ & (9w^2+2w+1)^{\gamma_2-1} - \frac{\omega_0H_0^2p_{1,2}^2}{E^2}a^{\xi_2}, \end{aligned} \quad (125)$$

where  $\xi_1 = -6(1+w)\gamma_2 + \frac{\gamma_1p_{1,2}}{E}$  and  $\xi_2 = -3(1+w) + \frac{2p_{1,2}}{E}$ .

## 2. Cosmological Constant

- Case-I:  $\omega(\phi) = \omega_0\phi^m$

If  $w = -1$ , then we have

$$H = H_0, \quad R = 12H_0^2,$$

then (123) transforms to

$$\begin{aligned} V_{1,2}(\phi) = & -\alpha_1\alpha_2(36)^{\gamma_2}H_0^{4\gamma_2}\phi^{\gamma_1} - \gamma_3\phi^{\gamma_4} - \gamma_5\phi^{\gamma_6} + \frac{1}{9}\alpha_1\alpha_2\gamma_2(36)^{\gamma_2}H_0^{4\gamma_2}\phi^{\gamma_1} - \frac{p_{1,2}}{E}\alpha_1\alpha_2\gamma_1\gamma_2(36)^{\gamma_2} \\ & \times H_0^{4\gamma_2}\phi^{\gamma_1} - \frac{\omega_0H_0^2p_{1,2}^2}{E^2}\phi^{m+2}. \end{aligned} \quad (126)$$

In terms of scale factor, we have

$$\begin{aligned} V_{1,2}(a) = & -\alpha_1\alpha_2(36)^{\gamma_2}H_0^{4\gamma_2}a^{\frac{\gamma_1p_{1,2}}{E}} - \gamma_3a^{\frac{\gamma_4p_{1,2}}{E}} - \gamma_5a^{\frac{\gamma_6p_{1,2}}{E}} + \frac{1}{9}\alpha_1\alpha_2\gamma_2(36)^{\gamma_2}H_0^{4\gamma_2}a^{\frac{\gamma_1p_{1,2}}{E}} - \frac{p_{1,2}}{E}\alpha_1 \\ & \times \alpha_2\gamma_1\gamma_2(36)^{\gamma_2}H_0^{4\gamma_2}a^{\frac{\gamma_1p_{1,2}}{E}} - \frac{\omega_0H_0^2p_{1,2}^2}{E^2}a^{\frac{(m+2)p_{1,2}}{E}}. \end{aligned} \quad (127)$$

- Case-II:  $\omega(\phi) = \omega_0$

For constant coupling, we have

$$\begin{aligned} V_{1,2}(a) = & -\alpha_1\alpha_2(36)^{\gamma_2}H_0^{4\gamma_2}a^{\frac{\gamma_1p_{1,2}}{E}} - \gamma_3a^{\frac{\gamma_4p_{1,2}}{E}} - \gamma_5a^{\frac{\gamma_6p_{1,2}}{E}} + \frac{1}{9}\alpha_1\alpha_2\gamma_2(36)^{\gamma_2}H_0^{4\gamma_2}a^{\frac{\gamma_1p_{1,2}}{E}} - \frac{p_{1,2}}{E}\alpha_1 \\ & \times \alpha_2\gamma_1\gamma_2(36)^{\gamma_2}H_0^{4\gamma_2}a^{\frac{\gamma_1p_{1,2}}{E}} - \frac{\omega_0H_0^2p_{1,2}^2}{E^2}a^{\frac{2p_{1,2}}{E}}. \end{aligned} \quad (128)$$

### 3. Chaplygin Gas

Here we discuss the scalar potential in the presence of Chaplygin gas given by Eqs.(98) and (99).

- Case-I:  $\omega(\phi) = \omega_0 \phi^m$

Using (99), we have (110) of the form

$$\begin{aligned}
 V_{1,2} = & -\alpha_1 \alpha_2 \phi^{\gamma_1} (9/2)^{\gamma_2} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-\gamma_2} \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2} - \gamma_3 \phi^{\gamma_4} - \gamma_5 \phi^{\gamma_6} - 2\alpha_1 \alpha_2 \gamma_2 \phi^{\gamma_1} (9/2)^{\gamma_2-1} \\
 & \left[ -2\hat{A} - \frac{32\hat{B}}{a^6} + \frac{99\hat{B}^2}{2a^{12}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-1} - \frac{279\hat{B}^3}{8a^{18}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-2} \right] \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1-\gamma_2} \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2-1} \\
 & - 72\alpha_1 \alpha_2 \gamma_2 (\gamma_2 - 1) (9/2)^{\gamma_2-2} \left[ \frac{108\hat{A}\hat{B}}{a^6} - \frac{243\hat{B}^2}{a^{12}} + \frac{729\hat{B}^3}{2a^{18}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-1} - \frac{243\hat{B}^4}{2a^{24}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-2} \right] \\
 & \times \phi^{\gamma_1} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{2-\gamma_2} \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2-2} - \frac{36p_{1,2}}{E} \alpha_1 \alpha_2 \gamma_1 \gamma_2 \phi^{\gamma_1} (9/2)^{\gamma_2-1} \left( \hat{A} - \frac{2\hat{B}}{a^6} \right) \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1-\gamma_2} \\
 & \times \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2-1} - \frac{\omega_0 p_{1,2}^2}{\hat{E}^2} \phi^{m+2} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1/2}. \tag{129}
 \end{aligned}$$

In terms of scale factor, we have

$$\begin{aligned}
 V_{1,2}(a) = & -\alpha_1 \alpha_2 a^{\frac{\gamma_1 p_{1,2}}{E}} (9/2)^{\gamma_2} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-\gamma_2} \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2} - \gamma_3 a^{\frac{\gamma_4 p_{1,2}}{E}} - \gamma_5 a^{\frac{\gamma_6 p_{1,2}}{E}} - 2\alpha_1 \alpha_2 \gamma_2 \\
 & a^{\frac{\gamma_1 p_{1,2}}{E}} (9/2)^{\gamma_2-1} \left[ -2\hat{A} - \frac{32\hat{B}}{a^6} + \frac{99\hat{B}^2}{2a^{12}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-1} - \frac{279\hat{B}^3}{8a^{18}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-2} \right] \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1-\gamma_2} \\
 & \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2-1} - 72\alpha_1 \alpha_2 \gamma_2 (\gamma_2 - 1) (9/2)^{\gamma_2-2} \left[ \frac{108\hat{A}\hat{B}}{a^6} - \frac{243\hat{B}^2}{a^{12}} + \frac{729\hat{B}^3}{2a^{18}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-1} \right. \\
 & \left. - \frac{243\hat{B}^4}{2a^{24}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-2} \right] a^{\frac{\gamma_1 p_{1,2}}{E}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{2-\gamma_2} \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2-2} - \frac{36p_{1,2}}{E} \alpha_1 \alpha_2 \gamma_1 \gamma_2 a^{\frac{\gamma_1 p_{1,2}}{E}} \\
 & (9/2)^{\gamma_2-1} \left( \hat{A} - \frac{2\hat{B}}{a^6} \right) \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1-\gamma_2} \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2-1} - \frac{\omega_0 p_{1,2}^2}{\hat{E}^2} \phi^{(m+2)p_{1,2}/E} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1/2} \tag{130}
 \end{aligned}$$

- Case-II:  $\omega(\phi) = \omega_0$

For constant coupling, the potential takes the following form

$$\begin{aligned}
V_{1,2}(a) = & -\alpha_1\alpha_2 a^{\frac{\gamma_1 p_{1,2}}{E}} (9/2)^{\gamma_2} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-\gamma_2} \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2} - \gamma_3 a^{\frac{\gamma_4 p_{1,2}}{E}} - \gamma_5 a^{\frac{\gamma_6 p_{1,2}}{E}} - 2\alpha_1\alpha_2\gamma_2 \\
& a^{\frac{\gamma_1 p_{1,2}}{E}} (9/2)^{\gamma_2-1} \left[ -2\hat{A} - \frac{32\hat{B}}{a^6} + \frac{99\hat{B}^2}{2a^{12}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-1} - \frac{279\hat{B}^3}{8a^{18}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-2} \right] \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1-\gamma_2} \\
& \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2-1} - 72\alpha_1\alpha_2\gamma_2(\gamma_2-1)(9/2)^{\gamma_2-2} \left[ \frac{108\hat{A}\hat{B}}{a^6} - \frac{243\hat{B}^2}{a^{12}} + \frac{729\hat{B}^3}{2a^{18}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-1} \right. \\
& \left. - \frac{243\hat{B}^4}{2a^{24}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{-2} \right] a^{\frac{\gamma_1 p_{1,2}}{E}} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{2-\gamma_2} \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2-2} - \frac{36p_{1,2}}{E} \alpha_1\alpha_2\gamma_1\gamma_2 a^{\frac{\gamma_1 p_{1,2}}{E}} \\
& (9/2)^{\gamma_2-1} \left( \hat{A} - \frac{2\hat{B}}{a^6} \right) \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1-\gamma_2} \left( 8\hat{A}^2 + \frac{\hat{B}^2}{a^{12}} \right)^{\gamma_2-1} - \frac{\omega_0 p_{1,2}^2}{\hat{E}^2} \phi^{2p_{1,2}/E} \left( \hat{A} + \frac{\hat{B}}{a^6} \right)^{1/2}. \quad (131)
\end{aligned}$$

We examine graphically the field potentials of more general de-Sitter model  $f(R, Y, \phi)$  given in (30) and power law models  $f(R, \phi)$ ,  $f(Y, \phi)$  for Chaplygin gas given in (101) and (130). We have plotted these field potentials versus scale factor “a” as shown in Figure 1 and 2. It can be seen that in case of de-Sitter model and power law  $f(Y, \phi)$  model, we have positive decreasing scalar field potential while in power law  $f(R, \phi)$  model, positive increasing scalar field potential with increasing scale factor. It can be concluded that to get acceptable positive field potential, we should choose negative values of  $\omega_0$  and positive value of  $\beta$ .



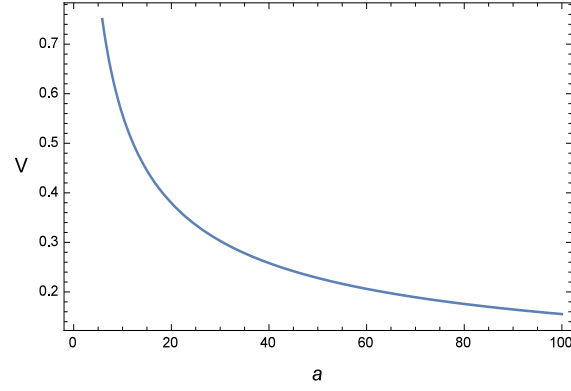


FIG. 1: Plot of field Potential versus scale factor “a” with  $\alpha_1 = -2 \times 10^{-10}$ ,  $\alpha_2 = -10^{-10}$ ,  $\alpha_3 = 3 \times 10^8$ ,  $n = 0.05$ ,  $\beta = 5 \times 10^{-5}$ ,  $\omega_0 = -5 \times 10^{-24}$  and  $H_0 = 67.3$  given by Eq. (30).

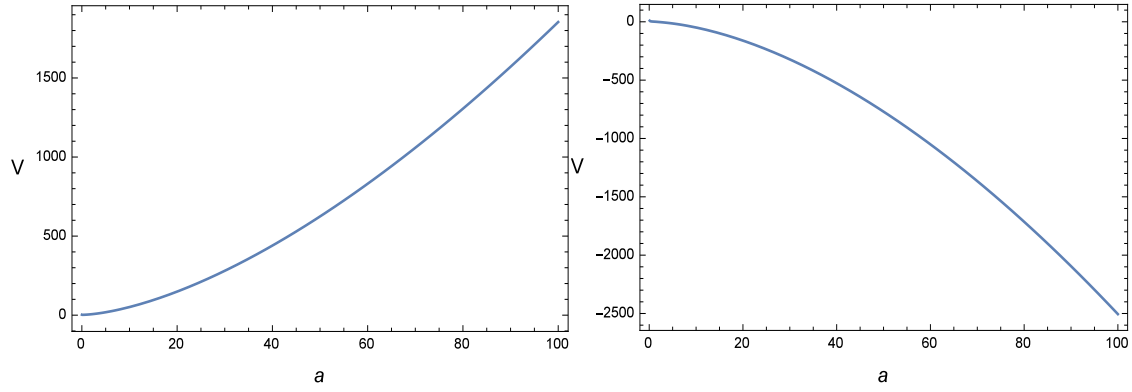


FIG. 2: Plots of field Potential versus scale factor “a” with  $\alpha_1 = -0.2$ ,  $\alpha_2 = 1$ ,  $n = 1.5$ ,  $\beta = 0.5$  and  $\omega_0 = -2$  given in Eqs. (101) and (130).

## Appendix A

Eq. (9) in terms of “ $a$ ” can be written as

$$\begin{aligned}
& 6a^2 H^3 \phi'' f_{R\phi} + 12a^2 H^3 (3H^2 + 2aHH') \phi'' f_{Y\phi} + (24aH^3 + 18a^2 H^2 H') \phi' f_{R\phi} + (2a^4 H^4 H''' + 38 \\
& \times a^3 H^4 H'' + 8a^4 H^3 H' H'' + 214a^2 H^4 H' + 88a^3 H^3 H'^2 + 2a^4 H^2 H'^3 + 32aH^5) \phi' f_{Y\phi} - 36aH^2 \times \\
& (a^2 H^2 H'' + a^2 HH'^2 + 5aH^2 H') \phi' f_{RR\phi} + 36aH^2 (2a^2 H^4 H'' + 2a^3 H^3 H' H'' + 2aH^4 H' + 4a^2 H^3 \\
& \times H'^2 + 2a^3 H^2 H'^3) \phi' f_{RY\phi} + 72aH^2 (24a^2 H^6 H'' + 34a^3 H^5 H' H'' + 12a^4 H^4 H'^2 H'' + 96aH^6 H' \\
& + 154a^2 H^5 H'^2 + 78a^3 H^4 H'^3 + 12a^4 H^3 H'^4) \phi' f_{YY\phi} + 6a^2 H^3 \phi'^2 f_{R\phi\phi} - 6a^2 H^3 \omega(\phi) \phi'^2 + 12a^2 H^3 \\
& \times (2aHH' + 3H^2) \phi'^2 f_{Y\phi\phi} + 6 (7aH^2 H' + a^2 HH'^2 + a^2 H^2 H'') f_R + 2 (a^4 H^4 H'''' + 10a^3 H^4 H''' \\
& + 7a^4 H^3 H' H''' + 25a^2 H^4 H'' + 49a^3 H^3 H' H'' + 11a^4 H^2 H'^2 H'' + 4a^4 H^3 H''^2 + 3aH^4 H' + 49a^2 \times \\
& H^3 H'^2 + 19a^3 H^2 H'^3 + a^4 HH'^4) f_Y - 36 (a^3 H^4 H''' + 7a^2 H^4 H'' + 5a^3 H^3 H' H'' + 5aH^4 H' + 17 \\
& \times a^2 H^3 H'^2 + 2a^3 H^2 H'^3) f_{RR} + 36 (2a^3 H^6 H''' + 2a^4 H^5 H' H''' + 6a^2 H^6 H'' + 24a^3 H^5 H' H'' + 2a^4 \\
& \times H^5 H''^2 + 14a^4 H^4 H'^2 H'' + 2aH^6 H' + 18a^2 H^5 H'^2 + 22a^3 H^4 H'^3 + 6a^4 H^3 H'^4) f_{RY} + 72 (24a^3 \\
& \times H^8 H''' + 34a^4 H^7 H' H''' + 12a^5 H^6 H'^2 H''' + 144a^2 H^8 H'' + 34a^4 H^7 H''^2 + 486a^4 H^6 H'^2 H'' + 578 \\
& \times a^3 H^7 H' H'' + 24a^5 H^6 H' H''^2 + 108a^5 H^5 H'^3 H'' + 438a^4 H^5 H'^4 + 48a^5 H^4 H'^5 + 1158a^3 H^6 H'^3 \\
& + 980a^2 H^7 H'^2 + 96aH^8 H') f_{YY} = 0
\end{aligned} \tag{A.1}$$

## V. CONCLUSIONS

In this paper, we have examined the scalar field potentials by a procedure known as reconstruction of field potentials. We have used flat FRW model in a well-known general scalar tensor gravity. We have derived the general form of field potential without using any specific value of  $f$ ,  $V$  and  $H$ . In this paper, we have investigated the field potentials by taking three de-Sitter and two power law models which were constructed in [54]. We have taken  $\omega = \omega_0 \phi^m$  in all cases and field potential is based on scale factor “ $a$ ”, scalar field  $\phi$  and Hubble parameter  $H$ . It is noticed that without choosing any specific value of Hubble parameter, it is impossible to obtain the explicit form of field potential. For this reason, we consider the Hubble parameter for barotropic fluid, the cosmological constant and the Chaplygin gas matter contents in separate cases. In literature, the scalar field potentials which usually studied are positive and inverse power laws, exponential and logarithmic potentials [61, 62], whereas others are different combinations of these functions.

In de-Sitter models, we have found the form of potential by reconstruction in terms of scalar

field. We have found the scale factor in terms of scalar field with negative power in all cases of de-Sitter and in case of power law models, we cannot get the explicit potential but using barotropic fluid, cosmological constant and Chaplygin gas matter content with  $\omega(\phi) = \omega_0 \phi^m$ , we have found the explicit form of scalar field potential.

Graphically we have showed here just three plots, de-Sitter  $f(R, Y, \phi)$  model, power law  $f(R, \phi)$  and  $f(Y, \phi)$  models for Chaplygin gas matter content. In de-Sitter  $f(R, Y, \phi)$  model the two cases  $\alpha_1 > 0, \alpha_2 > 0$  and  $\alpha_1 > 0, \alpha_2 < 0$  we have positive decreasing scalar field if  $\alpha_3$  and  $\omega_0 > 0$  have same sign and positive increasing scalar field if  $\alpha_3$  and  $\omega_0 > 0$  have opposite sign. In other two cases:  $\alpha_1 < 0, \alpha_2 > 0$  and  $\alpha_1 < 0, \alpha_2 < 0$ , we have positive increasing scalar field if  $\alpha_3$  and  $\omega_0 > 0$  have same sign and positive decreasing scalar field if  $\alpha_3$  and  $\omega_0 > 0$  have opposite sign.

Observing power law models, we have noticed that we have four cases:  $\alpha_1 \alpha_2$  both as positive, having opposite signs and both are negative. In  $f(R, \phi)$  model, we have positive increasing scalar field potential for all cases if  $\omega_0 < 0$  and negative decreasing scalar field potential if  $\omega_0 > 0$ . In  $f(Y, \phi)$  model, two cases having  $\alpha_2 > 0$  we have negative decreasing field potential for  $\omega_0 < 0$  and for  $\omega > 0$  plot has signature flip from negative to positive if  $\alpha_1 > 0$  and signature flip from positive to negative if  $\alpha_1 < 0$ . Other two cases having  $\alpha_2 < 0$ , for  $\omega_0 > 0$  we have positive increasing field potential and signature flip from positive to negative if  $\alpha_1 < 0$  and for  $\alpha_1 > 0$  signature flip from negative to positive.

In [30] author has used the scalar-tensor gravity with flat FLRW and used induced gravity to reconstruct the scalar field potential. Further used barotropic fluid, cosmological constant, Chaplygin gas and modified Chaplygin gas to reconstruct the Scalar field potential. In same theory, Sharif and Saira [52] has used Bianchi-I universe model and induced gravity to reconstruct the scalar field potential. Further they also used barotropic fluid, cosmological constant and Chaplygin gas to reconstruct the Scalar field potential. Now in this paper, we are working on more extended scalar tensor theory with FLRW universe, used some de-Sitter and power law models to reconstruct the scalar field potential, de-Sitter models are not used before to reconstruct the scalar field potential. We also used barotropic fluid, cosmological constant and Chaplygin gas to reconstruct the Scalar field potential. In Lagrangian, if we choose  $f(R, Y, \phi) = Rf(\phi), \omega(\phi) = -1/2$ , we get same Friedman equations, Klein-Gordon equation and scalar field potential constructed in [30]. If in power law  $f(R, \phi)$  model, we choose parameters  $\alpha_1 = 1/2, \alpha_2 = \gamma, \gamma_1 = 2, \gamma_2 = 1, \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 0$ , we get the same formula of induced gravity and from this, we can get the same results.

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